Local and Global Stability of Delayed Congestion Control Systems

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***Abstract*—Stability proofs of nonlinear congestion control systems under heterogeneous feedback delays are usually difficult and involve a fair amount of effort. In this paper, we show that there exist a class of congestion control methods that admit very simple proofs of asymptotic stability and allow control equations to be *delay-independent*. This is in contrast to most previous work, which requires that each flow (and sometimes each router) adapt its control-loop constants based on the feedback delay and/or the length of the corresponding end-to-end path. Our new congestion control method, which we call *Max–Min Kelly Control* (MKC), builds upon Kelly's original work in [3] and allows end-flows to be stable and fair regardless of network feedback delays or the number of hops in their end-to-end paths. Using basic matrix algebra and discrete control theory, we show MKC's local asymptotic stability under heterogeneous, directional feedback delays. We also offer a simple proof of its global asymptotic stability assuming constant feedback delay.**

***Index Terms*—Asymptotic stability, congestion control, delay.**

I. Introduction

Recent research efforts [2]–[9], [11], [12] offer an innovative interpretation of Internet congestion control mechanisms from the perspective of economics. One representative method in this category is the framework originated by Kelly *et al.* [3], in which both optimization and game theory are used to model the network and the end-flows. Kelly's approach models end-users as distributed and noncooperative entities, where each entity implements an independent strategy to maximize its locally maintained utility and minimize the prices paid for using network resources. Stimulated by Kelly's work, subsequent studies [2], [9], [11], [12] and various extensions [4]–[8] of Kelly controls have formed a distinct research area inside current Internet congestion control.

Most current studies [3], [5]–[8], [10]–[12] of Kelly controls are conducted on the basis of a *continuous*-time fluid model; however, all real networks are discrete and thus may exhibit different stability conditions from those derived using continuous fluid models. Moreover, local stability conditions derived in prior work [2], [9], [11], [12] require that parameters of the control equation be adaptively tuned according to feedback delays D_i, which is undesirable in practice since it leads to unfairness between the flows and oscillations when the delays are not properly estimated by the users. Finally, prior work typically assumes that $D_i(t) = D_i$ is constant over time, which leads to uncertainty as to whether (and how) their stability conditions hold under random (stochastic) delay $D_i(t)$ often found in the real Internet.

To overcome these limitations, this paper provides a new insight into discrete Kelly controls and demonstrates how to stabilize them under random (heterogeneous) feedback delays and *constant* gain parameters of the control equation. We accomplish this task through a simple modification to the control loop of Kelly's controller and offer a fresh look at this framework by associating it with max-min fairness instead of the original *proportional fairness*[3]. Accordingly, we call this new controller *Max–Min Kelly Control* (MKC) and demonstrate that it is both *locally* asymptotically stable regardless of feedback delays (which can be random or otherwise) and *globally* asymptotically stable under constant delay D.

The rest of this paper is organized as follows. In Section II, we construct the system model and clarify the assumptions used throughout the paper. In Section III, we discuss the details of MKC. In Section IV, we present several generic results on stability of delayed systems and prove local asymptotic stability of MKC under heterogeneous delay. In Section V, we examine MKC's global asymptotic stability under constant feedback delay. In Section VI, we conclude our work and suggest directions for future research.

II. Modeling Assumptions and Previous Work

A. Delayed Congestion Control Systems

Assume that a network system consists of N users, M resources, and a certain number of data links that connect all these components. Each user i is identified by a route r_i. All network resources (routers) continuously send congestion feedback to those users in whose path they appear. Using this feedback information, each source updates its sending rate according to some control equation with the goal to maintain a fair and oscillation-free sharing of network resources.

We next describe how directional and heterogeneous feedback delays are introduced in the control loop. Delays in network feedback arise from both the transmission/propagation time along the data links and the queuing delays at each of the intermediate routers. Consider a scenario where routers j and k are on the path of user i. The time lag for a packet to travel from sender i to router j is denoted by forward delay $D_{ij}^{\rightarrow}$, while the delay from router j to the receiver and subsequently from the receiver back to the sender is denoted by backward delay $D_{ij}^{\leftarrow}$. It is clear that the sum of directional delays with respect to each router is the round trip delay of user i, i.e., $D_i = D_{ij}^{\rightarrow} + D_{ij}^{\leftarrow} = D_{ik}^{\rightarrow} + D_{ik}^{\leftarrow}$.

Under this framework, we next review the class of utility-based controllers proposed in [3] and investigate their delayed stability in the rest of this paper.

B. Related Work

Recall that the classic Kelly control implements the primal algorithm of the network optimization problem described in [3]. In contrast to the significant research effort [3], [5]–[8], [10]–[12] put into the continuous-time analysis of Kelly control, Johari *et al.* introduced a discrete-time version of Kelly's control equation [2]

$$x_i(n) = x_i(n-1) + \kappa_i \left(\omega_i - x(n - D_i) \sum_{j \in r_i} \mu_j (n - D_{ij}^{\leftarrow}) \right) \quad (1)$$

where κ_i is a strictly positive gain parameter and ω_i is interpreted as the willingness of user i to pay the price for using the network. According to this notation, $D_i = 1$ means instantaneous (i.e., most-recent) feedback and $D_i \geq 2$ means delayed feedback. In (1), $\mu_j(n)$ is the congestion indication function of resource j, which is given by $\mu_j(n) = p_j \left(\sum_{u: j \in r_u} x_u (n - D_{uj}^{\rightarrow}) \right)$, where $p_j(\cdot)$ is the price charged by resource j. Notice that p_j depends only on the *combined* rate of all flows passing through router j at time n.

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Recall that for a *fixed* feedback delay D, system (1) is locally asymptotically stable if [2]

$$\kappa_i \sum_{j\in r_i} \left(\left(p_j + p_j' \sum_{u:j\in r_u} x_u \right) \Bigg|_{x_u^*} \right) < 2\sin\left(\frac{\pi}{2(2D-1)}\right) \quad (2)$$

where x_u^* is the stationary point of user u and $p_j(\cdot)$ is assumed to be differentiable at x_u^*. Additionally, Massoulié and Vinnicombe investigate Kelly's framework under heterogeneous feedback delays based on a continuous-time fluid model and derive sufficient stability conditions, which refine the upper bound in (2) to $1/D_i$ [9] and $\pi/(2D_i)$ [11], respectively. However, the analysis of discrete stability under heterogeneous delays or the analytical understanding of global stability are missing from the current picture.

Note that all recent studies and proposed controllers [2], [9], [11], [12] based on Kelly's framework require that end-users adapt their parameter κ_i *inverse proportionally* to D_i (observe in (2) that the right-hand side tends to zero for large D). Forcing the end-flows to keep $\kappa_i \sim 1/D_i$ leads to problems since users are often not aware of the increase in their delay *until after* oscillations have started and are not able in practice to properly adjust gain parameter κ_i in response to such increases in D_i. Even if we assume that each user can track delay D_i and keep κ_i normalized by D_i, the resulting system becomes unfair and favors users with smaller RTTs.

We should also note that Ying *et al.* [13] recently established delay-independent stability conditions for a family of utility functions and a generalized controller (1). Their work is similar in spirit to ours; however, the analysis and proposed methods are different.

III. MAX–MIN KELLY CONTROL (MKC)

To improve the practical aspects of discrete Kelly controls and decouple the delay and path length from gain parameters, in [14], we proposed a new discrete-time congestion control method based on several modifications to the classic Kelly control (1).

Our first change involves proper selection of the reference rate in (1), which currently applies feedback information about rate $x(n-D_i)$ to the most-recent rate $x(n-1)$. Our second improvement removes the dependency of stability on the number of resources along path r_i, which we accomplish by feeding back the packet loss from the *most congested* router in r_i. Thus, the end-user equation becomes

$$x_i(n) = x_i(n-D_i) + \alpha - \beta\eta_i(n)x_i(n-D_i) \quad (3)$$

where parameters $\alpha = \kappa_i\omega_i, \beta = \kappa_i$ are fixed for all users and $\eta_i(n)$ is the congestion indication function of user i: $\eta_i(n) = \max_{j\in r_i} p_j(n - D_{ij}^{\leftarrow})$. Here, $p_j(\cdot)$ is the packet loss of router j and depends on the aggregate input rate

$$p_j(n) = p_j\left(\sum_{u\in s_j} x_u(n - D_{uj}^{\rightarrow})\right) \quad (4)$$

where s_j is the set of users passing through router j. We call this new controller (3)–(4) MKC [14].

In particular, we can also specify $p_j(n)$ with the following standard packet loss function:

$$p_j(n) = \frac{\sum_{u\in s_j} x_u(n - D_{uj}^{\rightarrow}) - C_j}{\sum_{u\in s_j} x_u(n - D_{uj}^{\rightarrow})} \quad (5)$$

where C_j is the capacity of router j. Besides proving max-min fairness, (5) also allows "negative" packet-loss feedback when the bottleneck resource is under-utilized (i.e., the combined rate of all flows passing through the resource is less than its capacity). As we show later in the paper, this change improves the convergence rate to link utilization from linear to exponential. Hence, the resulting controller is called Exponential MKC (EMKC) [14].

In what follows in this paper, we seek to gain an in-depth understanding and provide analytical proofs of MKC's delayed stability in single-link scenarios. We start our investigation with its local properties in Section IV.

IV. DELAYED LOCAL STABILITY

A. Delayed Linear Stability

Before focusing on MKC, we first show the existence of a class of delayed control systems, whose stability directly follows from that of the corresponding undelayed systems, and later show that MKC falls into this category. Examine the following theorem that formalizes the generic law mentioned above. Due to limited space, we omit proofs in the section and refer interested readers to [14] for detail.

Theorem 1: Assume an undelayed linear system $\mathcal{L}$ with N flows

$$x_i(n) = \sum_{j=1}^{N} a_{ij}x_j(n-1). \quad (6)$$

If the coefficient matrix $A = (a_{ij})$ is real-valued and *symmetric*, then system $\mathcal{L}_D$ with arbitrary directional delays

$$x_i(n) = \sum_{j=1}^{N} a_{ij}x_j(n - D_j^{\rightarrow} - D_i^{\leftarrow}) \quad (7)$$

is asymptotically stable if and only if $\mathcal{L}$ is stable.

Theorem 1 opens an avenue for inferring stability of a delayed linear system based on the stability properties and coefficient matrix A of the corresponding undelayed system. Moreover, Theorem 1 is also applicable to nonlinear systems as we show in the following corollary.

Corollary 1: Assume an undelayed N-dimensional nonlinear system $\mathcal{N}$

$$x_i(n) = f_i\Big(x_1(n-1), x_2(n-1), \ldots, x_N(n-1)\Big) \quad (8)$$

where $\{f_i | f_i : \mathbb{R}^N \to \mathbb{R}\}$ is the set of nonlinear functions defining the system. If the Jacobian matrix J of this system is symmetric and real-valued, system $\mathcal{N}_D$ with arbitrary delay

$$x_i(n) = f_i\Big(x_1(n - D_1^{\rightarrow} - D_i^{\leftarrow}), x_2(n - D_2^{\rightarrow} - D_i^{\leftarrow}), \ldots, x_N(n - D_N^{\rightarrow} - D_i^{\leftarrow})\Big) \quad (9)$$

is locally asymptotically stable in the stationary point $\mathbf{x}^*$ if and only if $\mathcal{N}$ is stable in $\mathbf{x}^*$.

Based on the principle demonstrated above, we next examine local stability of MKC under random (heterogeneous) feedback delays.

B. Local Asymptotic Stability of MKC

We first derive the condition of local asymptotic stability of MKC (3)–(4), whose feedback generating function $p(n)$ is assumeed to be differentiable in the stationary point and has the same first-order partial derivative for all end-users. Following that, we will specialize this result to EMKC with the particular packet loss function (5).

We approach this problem by applying Theorem 1, whose first step is to show stability of the undelayed system

$$x_i(n) = (1 - \beta p(n-1))x_i(n-1) + \alpha \quad (10)$$

where $p(n)$ is the undelayed version of (4).

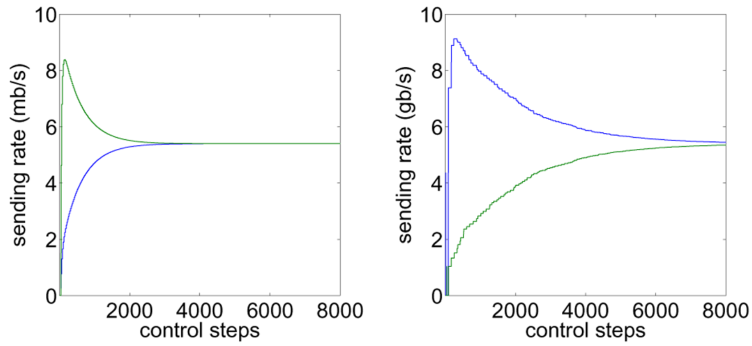

Fig. 1. Delayed behavior of MKC: (a) dynamics under constant delay $D = 20$ time units and (b) dynamics under delays randomly distributed between 1 and 100 time units.

Theorem 2: The undelayed N-dimensional MKC system is locally asymptotically stable if and only if

$$0 < \beta p^* < 2 \text{ and } 0 < \beta p^* + \beta N x^* \left.\frac{\partial p}{\partial x_i}\right|_{\mathbf{x}^*} < 2 \quad (11)$$

where x^* is the fixed point of each individual user, vector $\mathrm{x}^* = \langle x^*, x^*, \ldots, x^* \rangle$ is the fixed point of the entire system, and p^* is the stationary packet loss.

According to the proof of Theorem 2, Jacobian J of the undelayed system (10) evaluated in $\mathbf{x}^*$ is real-valued and symmetric. Thus, combining this observation with the result of Corollary 1, we obtain that heterogeneously delayed MKC is also locally asymptotically stable in $\mathbf{x}^*$.

Corollary 2: The heterogeneously delayed MKC system (3)–(4) is locally asymptotically stable if and only if (11) is satisfied.

Corollary 2 is a generic result that is applicable to MKC with a wide class of congestion-indicator functions $p(n)$. Note further that, for a given controller with pricing function $p(n)$, condition (11) is easy to verify and does *not* depend on feedback delays. This is in contrast to all current studies [2], [9], [11], [12], whose results are dependent on individual feedback delay D_i. We next extend the above analysis to EMKC with the particular feedback given in (5).

Theorem 3: The heterogeneously delayed EMKC system defined by (3) and (5) is locally asymptotically stable if and only if $0 < \beta < 2$.

To better understand the implication of this theorem, consider an illustration in Fig. 1, where two EMKC flows ($\alpha = 200$ mb/s and $\beta = 0.5$) share a bottleneck link of capacity 10 gb/s. Recall that for the same setup, the classic Kelly control is unstable for any delay $D > 3$ time units [14]. In the first example, the feedback delay is 20 time units for each flow, while in the second example, delays of each flow randomly fluctuate between 1 and 100 time units at each control step. As seen in both examples in Fig. 1, full link utilization is reached without oscillations and eventually the two flows share the resource fairly. These simulation results support our previous conclusion that EMKC is a stable and fair controller under *random* delays, which is a requirement for any practical method to be used in the current Internet.

V. Global Stability Under Constant Delay

Recall that global asymptotic stability of a nonlinear dynamic system requires both Lyapunov stability and global quasi-asymptotic stability (whose definition follows later) in the *unique* stable fixed point [1]. Note that we proved local asymptotic stability of EMKC in the preceding section, which implies Lyapunov stability of the system. Thus, our remaining task is to prove that EMKC will converge to the unique fixed point regardless of its initial conditions. To accomplish this, we first consider several auxiliary results.

A. Preliminaries

We start with a very simple lemma.

Lemma 1: For an arbitrary sequence v_n such that $v_n \to 0$ for $n \to \infty$ and another sequence α_n such that $\forall n > n_0\colon |\alpha_n| < 1 - \varepsilon$, where $\varepsilon > 0$, the following recurrence converges to zero regardless of the value of x_0: $x_n = \alpha_n x_{n-1} + v_n$.

Proof: Defining a new set of variables such that $y_n = x_{n+n_0}$, $\beta_n = \alpha_{n+n_0}$, and $u_n = v_{n+n_0}$ to shift recurrence x_n by n_0 time units forward and skipping the transient region of the evolution of x_n when α_n can potentially be larger than 1, we obtain $y_n = \beta_n y_{n-1} + u_n$. Using these assignments, $|\beta_n|$ is less than $1 - \varepsilon$ for all $n \geq 0$. We next demonstrate that sequence y_n converges to zero, which implies that x_n does too. Recursively expanding y_n, for $n \geq 2$, we get

$$y_n = \prod_{i=1}^{n} \beta_i y_0 + u_n + \sum_{i=1}^{n-1} u_i \prod_{j=i+1}^{n} \beta_j. \quad (12)$$

For convenience of presentation, let

$$S_1(n) = \prod_{i=1}^{n} \beta_i y_0 + u_n \text{ and } S_2(n) = \sum_{i=1}^{n-1} \left(u_i \prod_{j=i+1}^{n} \beta_j \right). \quad (13)$$

Since $|\beta_n| < 1 - \varepsilon$ and u_n is a time-shifted version of v_n, we immediately obtain that $S_1(n) \to 0$ as $n \to \infty$. Next examine $S_2(n)$ and show that it also tends to zero for large n. Re-writing (13)

$$|S_2(n)| \leq \sum_{i=1}^{n-1} \left(|u_i| \prod_{j=i+1}^{n} |\beta_j| \right). \quad (14)$$

Again since $|\beta_n| < 1 - \varepsilon$, we have

$$|S_2(n)| \leq \sum_{i=1}^{n-1} |u_i|(1-\varepsilon)^{n-i} = G_1(n) + G_2(n) \quad (15)$$

where we define

$$G_1(n) = \sum_{i=1}^{n/2} |u_i|(1-\varepsilon)^{n-i} \quad (16)$$

and

$$G_2(n) = \sum_{i=n/2+1}^{n-1} |u_i|(1-\varepsilon)^{n-i}. \quad (17)$$

To show that both $G_1(n)$ and $G_2(n)$ converge to zero, we need the following notations: $m_1(n) = \max(|u_1|, \ldots, |u_{n/2}|)$ and $m_2(n) = \max(|u_{n/2+1}|, \ldots, |u_{n-1}|)$. Then we have

$$\begin{aligned} G_1(n) &\leq m_1(n) \sum_{i=1}^{n/2} (1-\varepsilon)^{n-i} = m_1(n) \sum_{j=n/2}^{n-1} (1-\varepsilon)^j \\ &= m_1(n) \left(\sum_{j=0}^{n-1} (1-\varepsilon)^j - \sum_{j=0}^{n/2-1} (1-\varepsilon)^j \right) \\ &= m_1(n) \frac{(1-\varepsilon)^n - (1-\varepsilon)^{n/2}}{\varepsilon}. \end{aligned} \tag{18}$$

Since $m_1(n)$ is bounded and $0 < \varepsilon < 1$, $G_1(n) \to 0$. For $G_2(n)$, we have

$$\begin{aligned} G_2(n) &\leq m_2(n) \sum_{i=n/2+1}^{n-1} (1-\varepsilon)^{n-i} \\ &\leq m_2(n) \sum_{i=0}^{\infty} (1-\varepsilon)^i = \frac{m_2(n)}{\varepsilon}. \end{aligned} \tag{19}$$

Notice that since both $u_{n/2}$ and u_n converge to zero, then so must $m_2(n)$. Therefore, we get $G_2(n) \to 0$, which leads to $S_2(n) \to 0$ and hence $y_n \to 0$.

We next present our main result of this section.

Theorem 4: Assume a nonlinear system $x_n = f(x_{n-1}, y_{n-1})$, where function $f(x,y)$ is linear in both arguments, i.e., $f(x,y) = a + bx + cy + dxy$, for some constants $a - d$. Further assume that y_n converges to a stationary point y^* as $n \to \infty$ and form another system, which replaces y_n with y^* in system x_n: $\tilde{x}_n = f(\tilde{x}_{n-1}, y^*)$. Then, system x_n converges if and only if system $\tilde{x}_n$ converges, in which case the two stationary points are the same regardless of the initial points x_0 and $\tilde{x}_0$ in which each system is started: $\lim_{n\to\infty} |x_n - \tilde{x}_n| = 0$.

Proof: We again only prove the sufficient condition. The necessary condition follows by reversing the order of steps. First notice that system $\tilde{x}_n$ is stable (bounded) if and only if $|b + dy^*| < 1$. Next denote by Δx_n the absolute distance between the trajectories of the two systems at time n: $\Delta x_n = x_n - \tilde{x}_n$. Further let $\Delta y_n = y_n - y^*$ be the distance of y_n from its stationary point. Then we can write

$$\begin{aligned} \Delta x_{n+1} &= x_{n+1} - \tilde{x}_{n+1} = f(x_n, y_n) - f(\tilde{x}_n, y^*) \\ &= f(x_n, y_n) - f(\tilde{x}_n, y_n) + f(\tilde{x}_n, y_n) - f(\tilde{x}_n, y^*) \\ &= (b + dy_n)\Delta x_n + (c + d\tilde{x}_n)\Delta y_n. \end{aligned} \tag{20}$$

Next notice that (20) defines a recursive relationship on Δx_n

$$\Delta x_n = \alpha_n \Delta x_{n-1} + v_n \tag{21}$$

where $\alpha_n = b + dy_n$ and $v_n = (c + d\tilde{x}_n)\Delta y_n$. First, since $\tilde{x}_n$ is bounded and $\Delta y_n \to 0$ as $n \to \infty$, we have $v_n \to 0$ for large n. Second, since $|b+dy^*| < 1$, there exists such ε that: $|b+dy^*| < 1-2\varepsilon$.

Since $y_n \to y^*$, there exists such n_0 that $\forall n > n_0$, sequence α_n is bounded by the following:

$$|\alpha_n| = |b + dy_n| \langle 1 - \varepsilon, \forall n \rangle n_0. \tag{22}$$

Thus, system (21) satisfies the conditions of Lemma 1 and converges to zero as $n \to \infty$.

B. EMKC

We next show global stability of the combined rate $X(n)$ of N EMKC flows sharing a single bottleneck and convergence of loss $p(n)$ to p^* regardless of the behavior of flow rates $x_i(n)$.

Lemma 2: When $0 < \beta < 2$, the combined rate $X(n)$ of EMKC is globally asymptotically stable under constant delay and converges to $X^* = C + N\alpha/\beta$ at an exponential rate.

Proof: Assume that delay D is constant. Combining (3)–(5) and taking the summation for all N flows, we get that EMKC's combined rate $X(n) = \sum_i x_i(n)$ forms a linear system

$$\begin{aligned} X(n) &= \left(1 - \beta \frac{X(n-D) - C}{X(n-D)}\right) X(n-D) + N\alpha \\ &= (1-\beta)X(n-D) + \beta C + N\alpha. \end{aligned} \tag{23}$$

It is clear that the above linear system is stable if and only if $0 < \beta < 2$. Since convergence of linear systems implies global asymptotic stability, we can conclude that $X(n)$ is globally stable regardless of individual flow trajectories $x_i(n)$. We next show the convergence speed of $X(n)$. Recursively expanding (23), we have

$$X(n) = (1-\beta)^{n/D}(X_0 - X^*) + X^* \tag{24}$$

where X_0 is the combined initial rate and $X^* = C + N\alpha/\beta$ is the combined stationary rate of all flows. Notice that for $0 < \beta < 2$, the first term in (24) approaches zero exponentially fast and $X(n)$ indeed converges to X^*.

Using (5), it is not difficult to see that $p(n)$ can be expressed as $p(n) = 1 - C/X(n)$. Combining this observation with the result of Lemma 2, we immediately have the following corollary.

Corollary 3: When $0 < \beta < 2$, EMKC's packet loss $p(n)$ converges to $p^* = N\alpha/(C\beta + N\alpha)$ regardless of the initial rates of the flows or their individual rates $x_i(n)$.

Before showing global stability of EMKC, we first review the following stability concept that describes asymptotic properties of a dynamic system.

Definition 1: [1] A point $\mathbf{x}^*$ is *globally* quasi-asymptotically stable if and only if for all $\varepsilon > 0$ there exists n_0 such that for all $n > n_0$: $|\mathbf{x}(n) - \mathbf{x}^*| < \varepsilon$ regardless of the initial point $\mathbf{x}(0)$.

According to Corollary 1, EMKC is *locally* quasi-asymptotically stable in its unique fixed point $\mathbf{x}^*$. In what follows, we prove that each individual flow rate $x_i(n)$ is globally quasi-asymptotically stable, which implies that the entire system of flows $\mathbf{x}(n) = \langle x_1(n), \ldots, x_N(n) \rangle$ also exhibits global quasi-asymptotic stability.

Theorem 5: Assuming an N-flow EMKC system with constant delay D and an arbitrary initial point $\mathbf{x}(0) = \langle x_1(0), \ldots, x_N(0) \rangle$, $x_i(n)$ converges to $x^* = C/N + \alpha/\beta$ if and only if $0 < \beta < 2$.

Proof: We start with the sufficient condition. Under constant delay D, each EMKC flow activates a rate adjustment every D time units. Thus, we can define a new set of flows $\{u_i(t)\}$, which operate in time units scaled by a factor of D. Under this notation, we can write $x_i(n) = u_i(n/D) = u_i(t)$ and $x_i(n-D) = u_i(n/D - 1) = u_i(t-1)$. Notice that $u_i(t)$ has the same exact stability properties as $x_i(n)$. Select an arbitrary flow u_i and focus on its stability: $u_i(t) = f(u_i(t-1), p(t-1))$, where $p(t)$ is the packet loss at time t and $f(x,y)$ is given by

$$f(x,y) = (1 - \beta y)x + \alpha. \tag{25}$$

Then form a new system $\tilde{u}(t) = f(\tilde{u}(t-1), p^*) = (1-\beta p^*)\tilde{u}(t-1) + \alpha$, where $\tilde{u}(0) = u_i(0)$, and notice that the solution to this recurrence is stable if and only if $|b + dy^*| = |1 - \beta p^*| < 1$. This condition is automatically satisfied using the proof of EMKC's local stability in Theorem 3. According to Corollary 3, we notice that $p(t)$ converges to its unique stationary point p^* regardless of $\mathbf{x}(0)$. Since (25) is linear in each argument, we can apply Theorem 4 and immediately obtain that $u_i(n) \to \tilde{u}^* = C/N + \alpha/\beta$ and is therefore quasi-asymptotically stable regardless of the initial points $u_i(0)$ or $\mathbf{x}(0)$. Repeating the same argument for all flows i, we establish their individual convergence.

The necessity of condition $0 < \beta < 2$ directly follows from Corollary 3.

Combining EMKC's Lyapunov and global quasi-asymptotic stability, we have:

Corollary 4: EMKC is globally asymptotically stable under constant feedback delay D if and only if $0 < \beta < 2$.

VI. Conclusion

This paper offered a comprehensive stability analysis of a new congestion controller called MKC, which is proven to be locally stable with arbitrary (heterogeneous) feedback delays under easily verifiable conditions. This property makes MKC a highly appealing platform for congestion control in future high-speed networks with heterogeneous users. Moreover, we proposed a *negative* packet-loss feedback function to be used in conjunction with MKC and called the resulting controller EMKC. We proved that EMKC achieves both RTT-independent stability and fairness and converges to link utilization exponentially fast. Our investigation of global stability shows that all EMKC flows converge to their unique stationary points regardless of the initial point in which the system is started. We proved this fact for constant delays D and our future work is to extend the analysis to heterogenous delays.

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Controllability and Observability for a Class of Controlled Switching Impulsive Systems

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***Abstract*—In this note, we study the controllability and observability problem for a class of controlled switching impulsive systems. By proposing several formula of variation of parameters for this type of time-varying systems and employing the characteristic polynomial theory of matrix, we establish necessary and sufficient conditions for controllability and controlled observability with respect to a given switching time sequence. Specializing the obtained results to the case of time-invariant linear switching impulsive systems, we derive some simple algebraic criteria, which include the results reported in the literature for time-invariant linear switching systems, linear impulsive systems and classical linear systems. One example is worked out for illustration.**

***Index Terms*—Controllability, controlled observability, hybrid systems, switching impulsive systems, variation of parameters.**

I. Introduction

Motivated by the fact that hybrid systems provide a natural framework for mathematical modeling of many physical phenomena, their study has received considerable attention for the last two decades [1]–[3]. Most of the work encountered in the literature has focussed on two types of hybrid systems, namely; switching and impulsive systems. See [4]–[10] and the references cited therein for recent work on these two classes of systems. It is, however, worth noticing that switching and impulsive systems do not include some important hybrid systems existing in some applications characterized by switches of the states and abrupt changes at the switching instants.

Indeed, in many natural phenomena in systems such as evolutionary processes, biological neural networks and bursting rhythmic models in pathology, when certain quantities accumulate, the nature of the reaction undergoes an abrupt change. In this case, one needs to switch to a new system of differential equations taking into consideration a momentary perturbation of an impulsive nature. This class of systems exhibit simultaneously continuous-time dynamic switching and impulsive jump phenomena. A general description of these systems is called switching impulsive system. Examples include evolutionary processes and biological systems, as well as frequency-modulated signal processing systems, networked control systems, optimal control models in economics, and flying object motions. See [22]–[28].

The controllability and observability problem for hybrid systems has recently received considerable attention. See [11]–[18], [20], [21]. In [11], controllability and observability of periodic switching linear systems was studied. In [18] and [31], geometric criteria for controllability of switching systems was established. In [30], controllability of

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